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INHOMOGENEOUS INCLUSION IN AN ANISOTROPIC ELASTIC MEDIUM

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We solve the three-dimensional problem of the stress and strain distribution inside and on the surface of an inhomogeneous inclusion in an anisotropic elastic medium and the interaction of an inhomogeneous inclusion with an external field. A distinction is made between an inhomogeneity, inclusion, and inhomogeneous inclusion. An inhomogeneity is taken to mean a region inside which the plastic constants are different from those of the medium; an inclusion is constructed to be a region with the same elastic properties as the medium, but has undergone some changes and is thus a source of internal stresses in the medium; and inhomogeneous inclusion is taken to mean a region with the properties of an inhomogeneity and an inclusion, simultaneously.

Below, inhomogeneous inclusion means a region, filled by material under pressure, whose elastic properties are different from those of the medium. The pressure is modeled by a layer of bulk forces distributed along the boundary of the region. At the same time an external stress field acts on the medium.

1. We consider a three-dimensional boundless anisotropic elastic medium with a region V, which is filled under pressure with a material having different elastic properties than does the medium. The tensor $c^{\alpha\beta\lambda\mu}(x)$ of the elastic moduli of the medium with an inhomogeneous inclusion is written as

$$c(x) = c_0 + c_1 V(x), \tag{1}$$

where $x = (x_1, x_2, x_3)$ is a point of the medium; V(x) is a characteristic function of the region occupied by the inclusion ($V(x) = 1$, $x \in V$; $V(x) = 0$, $x \notin V$); c_0 is the tensor of the elastic constants of the homogeneous medium; and c_1 is a constant tensor characterizing the change in the elastic constants inside the inclusion. For a strip $c_1 = -c_0$ and for an absolutely rigid inclusion $c_1 \to \infty$. By $q^\alpha(x)$ we denote the layer of bulk forces distributed along the surface of the inhomogeneous inclusion and modeling the pressure under which the region was filled. We set

$$q_1^\alpha(x) = p^{\alpha\nu} n_\nu \delta(S). \tag{1.1}$$

Here $p^{\alpha\nu} = p^{\nu\alpha}$ is a given tensor; $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector normal to the surface S of the inhomogeneous inclusion; and $\delta(S)$ is the Dirac δ function, concentration on the surface.

Suppose that at the same time an external stress field $\sigma_0{}^{\alpha\beta}(x)$ acts on the medium. External, as usual, here means the field that would have been present in the homogeneous medium under the action of external forces $q_0{}^\alpha(x)$. The forces $\mathbf{q}_0$ are assumed not to contain a singularity of the type of a simple layer and a double layer. It is further assumed that the ordinary continuity condition for displacements $u_\alpha(x)$ and for the stress vector $\sigma_n{}^\alpha = \sigma^{\alpha\beta}\mathbf{n}_\beta$ is satisfied at the boundary of the inhomogeneous inclusion [1].

First we obtain integral equations for determining the strains inside the inhomogeneous inclusion. The displacements $u_\alpha(x)$ in the medium with an inhomogeneous inclusion satisfy the equations (written in operator form and constructed as generalized functions or distributions)

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$$-\nabla c \nabla u = q_0 + q_1,\ u(x) \to u_0(x) \quad \text{as} \quad x \to \infty$$

or

$$(L_0 + L_1)u = q_0 + q_1,\ L_0 = -\nabla c_0 \nabla,\ L_1 = -\nabla c_1 V \nabla, \tag{1.2}$$

where the operators L_0 and L_1 take the conditions at infinity into account; and $u_0(x)$ are the displacements in the homogeneous medium under the action of the forces q_0.

Suppose that $G_0 = L_0^{-1}$ is the Green's tensor for the homogeneous medium ($G_0L_0 = I$, I being the identity operator). Applying the operator def G_0 (the operator def corresponds to a symmetrized gradient) to both parts of Eq. (1.2), we have

$$\varepsilon - \mathrm{def} G_0 \nabla c_1 V \nabla u = \varepsilon_0 + \mathrm{def} G_0 q_1,$$
$$\varepsilon = \mathrm{def} u,\ \varepsilon_0 = \mathrm{def} u_0.$$

By the symmetry of the tensors of the elastic constants c_0 and c_1 these equations can be written as

$$\varepsilon + K_0 c_1 V \varepsilon = \varepsilon_0 + \mathrm{def} G_0 q_1,\ K_0 = -\mathrm{def} G_0 \mathrm{def}. \tag{1.3}$$

In detailed notation

$$\varepsilon_{\alpha\beta}(x) + \int_V K^0_{\alpha\beta\lambda\mu}(x - x') c_1^{\lambda\mu\kappa\rho} \varepsilon_{\kappa\rho}(x') dx' = \varepsilon^0_{\alpha\beta}(x) + \int_S \partial_{(\alpha} G^0_{\beta)\mu}(x - x') p^{\mu\nu}(x') n_\nu(x') dx'. \tag{1.4}$$

Here $G_{\alpha\beta}{}^0(x - x')$ is the kernel of the Green's operator G_0; $K_{\alpha\beta\lambda\mu}(x - x') = -[\partial_\alpha \partial_\lambda G_{\beta\mu}{}^0(x - x')]_{(\alpha\beta)(\lambda\mu)}$ is the kernel of the operator K_0; and (α, β) denotes symmetrization with respect to indices α and β.

We assume that the load is distributed uniformly over the surface of the inclusion. In this case the tensor $p^{\mu\nu}$ in Eq. (1.1) is constant and it can be shown that def $G_0 q_1 = K_0 V_p$. Indeed, taking the last integral in Eq. (1.4) over parts and taking into account the symmetry of the tensor p and the properties of the δ functions concentrated in the region and on the surface [2], we obtain

$$\mathrm{def} G_0 q_1 = \int_S \partial_{(\alpha} G^0_{\beta)\mu}(x - x') p^{\mu\nu} n_\nu(x') dx' =$$
$$- \int_V \partial_\nu \partial_{(\alpha} G^0_{\beta)\mu}(x - x') p^{\mu\nu} dx' = \int_V K^0_{\alpha\beta\lambda\mu}(x - x') p^{\mu\nu} dx' = K_0 V p.$$

Equations (1.3) now become

$$\varepsilon + K_0 c_1 V \varepsilon = \varepsilon_*,\ \varepsilon_* = \varepsilon_0 + \varepsilon_1 = \varepsilon_0 + K_0 V p.$$

Since by assumption the external forces q_0 do not contain singularities of the type of simple or double layers, the components of the strain $\varepsilon(x)$ on S are piecewise continuous. The equations obtained, therefore, are equivalent to the system

$$\varepsilon^+ + K_0^+ c_1 \varepsilon^+ = \varepsilon_*^+,\ \varepsilon_*^+ = \varepsilon_0^+ + K_0^+ p,\ \varepsilon^- = \varepsilon_0^- + K_0^-(p - c_1 \varepsilon^+), \tag{1.5}$$

where the first equation determines the deformation ε^+ inside the inhomogeneous inclusion and the second determines its continuation ε^- to the complement $\bar{V}$ of the region V; $K_0^+ = VK_0V$ is the constriction of the operator K_0 to the region V; $K_0^- = \bar{V}K_0V$. The operator solution of the equation has the form

$$\varepsilon^+ = (I + K_0^+ c_1)^{-1} \varepsilon_*^+$$

or in more symmetric form

$$\varepsilon^+ = (c_0 + c_0 K_0^+ c_1)^{-1} \sigma_*^+,\ \sigma_* = \sigma_0 + \sigma_1 = c_0 \varepsilon_0 + c_0 K_0^+ p. \tag{1.6}$$

Comparing the results with [1], we conclude that the problem of determining the strains inside an inhomogeneous inclusion in a medium acted upon an external field σ_0 is equivalent to the similar problem for an inhomogeneity acted upon by an external field $\sigma_* = \sigma_0 + \sigma_1$ ($\sigma_1 = c_0/K_0{}^+p$ is the field induced by the inclusion).

We note that the relation between an inclusion and an inhomogeneity was first obtained by Eshelby [3] for the case when an inclusion and an inhomogeneity are ellipsoidal, the medium and the inhomogeneity are isotropic, and the external field is uniform. The analogous problem for the anisotropic case was considered in [4].

Equations (1.6) generalize the results of [3, 4] and in closed form establish the reaction between an inhomogeneous inclusion of arbitrary shape in an anisotropic medium and an inhomogeneity for an arbitrary external field.

2. Suppose now that the region V occupied by the inhomogeneous inclusion is an ellipsoid with semiaxes a_1, a_2, a_3. In this case by virtue of its properties the operator $K_0{}^+$ [5] the inclusion-induced field $\sigma_1 = c_0K_0{}^+p$ is uniform and

$$\sigma_1 = c_0Ap,$$

where A is a constant tetravalent tensor, whose components are calculated in terms of the Fourier transform of the Green's tensor $G_0(\mathbf{n})$ of the medium. By $\langle f(\mathbf{n})\rangle$ we denote the mean value of the function f(n) over the ellipsoid:

$$\langle f(\mathrm{n})\rangle = \frac{a_1 a_2 a_3}{4\pi} \int_\Omega f(\mathrm{n})\rho^3(\mathrm{n})d\mathrm{n},$$
$$\rho(\mathrm{n}) = (a_1^2n_1^2 + a_2^2n_2^2 + a_3^2n_3^2)^{-1/2}. \tag{2.1}$$

Integration in $\langle f(\mathbf{n})\rangle$ is carried out over all directions of the unit vector $\mathbf{n}$, i.e., over the surface of the unit sphere. Then

$$A = \langle K_0(\mathrm{n})\rangle,\ K^0_{\alpha\beta\lambda\mu}(\mathrm{n}) = [n_\alpha G^0_{\beta\mu}(\mathrm{n})n_\lambda]_{(\alpha\beta)(\lambda\mu)},$$
$$G^0_{\beta\mu}(\mathrm{n}) = [c_0^{\alpha\beta\lambda\mu}n_\alpha n_\lambda]^{-1}. \tag{2.2}$$

The components of the tensors $G_0(\mathbf{n})$ and $K_0(\mathbf{n})$ can be calculated in explicit form for any anisotropic medium. They were obtained in [1, 6] for the particular cases of isotropic and orthotropic media. We note that the components of the tensor A and hence, the inclusion-induced field σ_1 depend on the geometric parameters of the region V. This dependence is concentrated in the scalar weighing factor $\rho(\mathbf{n})$ from Eq. (2.1), which simplifies the studies and the passages to the limit in the cases of a needle, a crack, and a disk.

We use the results obtained in [1] for an ellipsoidal inhomogeneity to determine the strains inside an ellipsoidal inhomogeneous inclusion.

First we consider the case when the external field σ_0 is uniform. Then the total field $\sigma_* = \sigma_0 + \sigma_1$ is also uniform. Since the external uniform field induces a uniform field of stress ε^+ inside V [3, 5, 7], from Eqs. (1.6) we find

$$\varepsilon^+ = (c_0 + c_0Ac_1)^{-1}\sigma_* = B^{-1}\sigma_*,$$
$$B = c_0 + c_0Ac_1 = \langle c_0 + c_0K_0(\mathrm{n})c_1\rangle = \langle B(\mathrm{n})\rangle. \tag{2.3}$$

As in the case of an inhomogeneity, the problem thus reduces to one of turning a constant tetravalent tensor B.

Suppose now that the external field σ_0 is linear. In [5, 8] the ellipsoidal region was shown to have polynomial conservatism: if the external stress field in the neighborhood of V is a polynomial of degree n, then the strain field induced inside V is a polynomial of the same degree. In particular field is a linear form in $\mathbf{x}$, then ε^+ is also a linear form.

We set $\sigma_0^{\alpha\beta} = d^{\alpha\beta}_{\cdot\cdot\lambda}x^\lambda$, where $d^{\alpha\beta}_{\cdot\cdot\lambda}$ is a given trivalent tensor. In this case the total field σ_* contains linear and uniform components σ_0 and σ_1. Since the first equation in (1.5) is linear in ε^+, it is equivalent to the system

$$\left.\begin{aligned}\varepsilon_1^+ + K_0^+c_1\varepsilon_1^+ &= \sigma_1\\ \varepsilon_2^+ + K_0^+c_1\varepsilon_2^+ &= \sigma_0\end{aligned}\right\}\ \varepsilon_1^+ + \varepsilon_2^+ = \varepsilon^+.$$

Here $\varepsilon_1{}^+$ is the strain caused by the uniform field σ_1 induced by the inclusion and $\varepsilon_2{}^+$ is the strain caused by the linear external field σ_0. The strain $\varepsilon_1{}^+$ is found from Eq. (2.3), where σ_* must be replaced by $\sigma_1 = c_0A_p$. In accordance with the property of polynomial conservatism of the ellipsoidal region, $\varepsilon_2{}^+$ can be sought in the form $(\varepsilon_2{}^+)_{\alpha\beta} = b_{\alpha\beta\lambda}x^\lambda$ where $b_{\alpha\beta\lambda}$ is an unknown trivalent tensor related to the given tensor $d_\lambda{}^{\alpha\beta}$ by

$$Da^2b = \frac{1}{3}d,\ D^{\alpha\beta\ \lambda\mu}_{\cdot\cdot\nu\cdot\cdot\eta} = \langle\rho(\mathrm{n})n_\nu B^{\alpha\beta\lambda\mu}(\mathrm{n})n_\eta\,\rho(\mathrm{n})\rangle.$$

Here $D^{\alpha\beta\cdot\lambda\mu}_{\cdot\nu\cdot\eta}$ is a constant hexavalent tensor, symmetric in pairs of indices $\alpha\beta$, $\lambda\mu$, $\nu\eta$; $a^2 = (a_i^2\delta_{ij})$ is a matrix of the third order, determined by the semiaxes of the ellipsoid a_i (i = 1, 2, 3); δ_{ij} is the Kronecker symbol; and B(**n**) is the tensor introduced in Eq. (2.1).

In the case of an external linear field the problem of determining the strains inside an inhomogeneous inclusion reduces to rotating two constant tensors, a tetravalent and a hexavalent tensor B and D, i.e., it reduces to a finite number of algebraic operations. We also emphasize that the solution of the problem was obtained in closed form, even though the Green's tensor of the anisotropic uniform medium is not known in explicit form.

If the strains $\varepsilon^{+}(\mathbf{x})$ inside the inhomogeneous inclusion are determined, the stresses $\sigma(\mathbf{n})$ on its outer surface under the given conditions at the boundary are equal to the stresses on the outer surface of an "equivalent" inhomogeneity and are calculated from [1]

$$\sigma^{\alpha\beta}(\mathrm{n}) = B^{\alpha\beta\lambda\mu}(\mathrm{n})\varepsilon^{+}_{\lambda\mu}.$$

In contrast to the case with an inhomogeneity, however, the study of the behavior of stresses on the surface of an inhomogeneous inclusion is complicated by the fact that the field σ_* depends on the geometric parameters of the region occupied by the inclusion.

3. Let us calculate the energy of the interaction of the inhomogeneous inclusion with the external field. We write the Green's operator G for a medium with an inhomogeneity in the form [9, 10]

$$G = G_0 - G_0\nabla P\nabla G_0, \tag{3.1}$$

where P is the interaction energy operator, whose kernel is concentrated in the region of inhomogeneity V and for which Kunin and Sosnina [9] obtained

$$P = -C_1(C_1 + C_1K_0^{+}C_1)^{-1}C_1,\ C_1 = c_1V. \tag{3.2}$$

If the operator C_1^{-1} is meaningful, P can be written as

$$P = -(C_1^{-1} + K_0^{+})^{-1}.$$

The total elastic energy of the system is

$$\Phi = \frac{1}{2}\int \mathrm{q(x)u(x)}d\mathrm{x} = \frac{1}{2}\int\int \mathrm{q(x)}G(\mathrm{x, x'})\mathrm{q(x')}d\mathrm{x}d\mathrm{x'}$$

or in the operator form

$$\Phi = \frac{1}{2}\mathrm{q}G\mathrm{q},\ \mathrm{q} = \mathrm{q}_0 + \mathrm{q}_1.$$

In Φ we replace G by the expression for it from (3.1) and write the total energy Φ as two terms

$$\Phi = \Phi_0 + \Phi_{\mathrm{int}}.$$

Here Φ_0 is the self-energy of the total field induced by the forces q_0 and q_1 in the homogeneous medium; Φ_{int} is the energy of the interaction of the inhomogeneous inclusion with that total field,

$$\Phi_0 = \frac{1}{2}\mathrm{q}G_0\mathrm{q},$$

$$\Phi_{\mathrm{int}} = -\frac{1}{2}\mathrm{q}G_0\nabla P\nabla G_0\mathrm{q} = \frac{1}{2}\varepsilon_*P\varepsilon_* = \frac{1}{2}\int_{V}\int_{V'} \varepsilon_*(\mathrm{y})P(\mathrm{y, y'})\varepsilon_*(\mathrm{y'})d\mathrm{y}d\mathrm{y'}.$$

We calculate the interaction energy Φ_{int} by the method proposed in [9] on the basis of the expansion of the kernel P(y, y') of the operator P from Eq. (3.2) in a series in multipoles. Suppose that the field q_0 and, therefore, ε_* are uniform. Then for Φ_* we have the exact formula

$$\Phi_{\mathrm{int}} = \frac{1}{2}\varepsilon^{*}_{\alpha\beta}P_0^{\alpha\beta\lambda\mu}\varepsilon^{*}_{\lambda\mu},$$

where P_0 is already a constant tetravalent tensor, the main term of the expansion of P(y, y'). If the inclusion is ellipsoidal, then

$$P_0 = -vc_1(c_1 + c_1Ac_1)^{-1}c_1 = -v(c_1^{-1} + A)^{-1}$$

(v is the volume of the ellipsoid and A is the constant tensor given in Eq. (2.2)).

In summary, for an ellipsoidal inhomogeneous inclusion in an external uniform field the problem is solved in explicit form for any anisotropic medium and anisotropic inclusion.

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